Appendix II – Derivation of the Boer-Lambert-Taylor (BLT) Diagram

The BLT (Boer Lambert Taylor) is a method to represent the space-time mean square difference between a model simulation and observations.

II.1.Definitions

Given two time series, *M=M(x,y,t)*, associated with a model simulation or a model-derived scenario and *A=A(x,y,t)* associated with observations or analyses, we can define the first and second order statistics in the following way.

The time-average of any of these variables can be written as

$$\overline{M(x,y)} = \frac{1}{N_p}\sum_{p=1}^{N_p} M(x,y,t_p) \tag{II.1}$$

while the space-average may be defined as

$$\langle M\rangle(t) = \frac{1}{N_i N_j}\sum_{i=1}^{N_i}\sum_{j=1}^{N_j} M(x_i,y_j,t) \tag{II.2}$$

Hence, the space-time-average takes the form

$$\langle\overline{M}\rangle = \frac{1}{N_i N_j N_p}\sum_{i=1}^{N_i}\sum_{j=1}^{N_j}\sum_{p=1}^{N_p} M(x_i,y_j,t_p). \tag{II.3}$$

We can define the second order statistics in the following way.

	Time	Space	Space-time
Anomaly	$M' = M - \overline{M}$ (II.4)	$M^* = M - \langle M\rangle$ (II.9)	$M^\circ = M - \langle\overline{M}\rangle$ (II.14)
Variance	$s'^2_M = \overline{(M - \overline{M})^2}$ (II.5)	$s^{*2}_M = \langle (M - \langle M\rangle)^2\rangle$ (II.10)	$s^{o2}_M = \langle\overline{(M - \langle\overline{M}\rangle)^2}\rangle$ (II.15)
Standard-deviation	$s'_M = \sqrt{\overline{(M - \overline{M})^2}}$ (II.6)	$s^*_M = \sqrt{\langle (M - \langle M\rangle)^2\rangle}$ (II.11)	$s^o_M = \sqrt{\langle\overline{(M - \langle\overline{M}\rangle)^2}\rangle}$ (II.16)
Correlation	$R'_{MA} = \frac{\overline{(M - \overline{M})(A - \overline{A})}}{s'_M s'_A}$ (II.7)	$R^*_{MA} = \frac{\langle (M - \langle M\rangle)(A - \langle A\rangle)\rangle}{s^*_M s^*_A}$ (II.12)	$R^o_{MA} = \frac{\langle\overline{(M - \langle\overline{M}\rangle)(A - \langle\overline{A}\rangle)}\rangle}{s^o_M s^o_A}$ (II.17)
Mean square difference	$\overline{d^2_{MA}} = \overline{(M - A)^2}$ (II.8)	$\langle d^2_{MA}\rangle = \langle (M - A)^2\rangle$ (II.13)	$\langle\overline{d^2_{MA}}\rangle = \langle\overline{(M - A)^2}\rangle$ (II.18)

II.2. Development of the space-time mean square difference for representation in a Taylor diagram

Using the space-time anomaly defined in (II.14) for M and A we obtain

$$M = \left\langle \overline{M} \right\rangle + M^\circ$$

and

$$A = \left\langle \overline{A} \right\rangle + A^\circ$$

Using these expressions in (II.18) yields

$$\left\langle \overline{d^2_{MA}} \right\rangle = \left\langle \overline{\left[\left(\left\langle \overline{M} \right\rangle - \left\langle \overline{A} \right\rangle\right) + (M^\circ - A^\circ)\right]^2} \right\rangle \tag{II.19}$$

By rearranging the terms, (II.19) can be written as

$$\left\langle \overline{d^2_{MA}} \right\rangle = \left\langle \overline{\left(\left\langle \overline{M} \right\rangle - \left\langle \overline{A} \right\rangle\right)^2} \right\rangle + \left\langle \overline{(M^\circ - A^\circ)^2} \right\rangle \tag{II.20}$$

and following the notation used in (II.18), equation (II.20) becomes

$$\left\langle \overline{d^2_{MA}} \right\rangle = \left\langle \overline{d^2_{\langle\overline{M}\rangle\langle\overline{A}\rangle}} \right\rangle + \left\langle \overline{d^2_{M^\circ A^\circ}} \right\rangle \tag{II.21}$$

By developing the second term of (II.21) we obtain

$$\left\langle \overline{d^2_{MA}} \right\rangle = \left\langle \overline{d^2_{\langle\overline{M}\rangle\langle\overline{A}\rangle}} \right\rangle + \left\langle \overline{M^{\circ 2}} \right\rangle - 2\left\langle \overline{M^\circ A^\circ} \right\rangle + \left\langle \overline{A^{\circ 2}} \right\rangle \tag{II.22}$$

that can be rewritten using (II.15) and (II.17)

$$\left\langle \overline{d^2_{MA}} \right\rangle = \left\langle \overline{d^2_{\langle\overline{M}\rangle\langle\overline{A}\rangle}} \right\rangle + s^{o2}_{M} + s^{o2}_{A} - 2 s^{o}_{M} s^{o}_{A} R^{o}_{MA} \tag{II.23}$$

In a Taylor diagram, the mean square difference associated with the spatial average of the stationary terms is usually not shown. Then the equation is scaled by the variance of the observations so that

$$\frac{\left\langle \overline{d^2_{MA}} \right\rangle_{Taylor}}{s^{o2}_{A}} = 1 + \frac{s^{o2}_{M}}{s^{o2}_{A}} - \frac{2 s^{o}_{M} R^{o}_{MA}}{s^{o}_{A}} \tag{II.24}$$

II.3. Development of the space-time mean square difference for representation in a BLT diagram.

It can be shown that the space-time variances can take the form

$$s_M^{o2} \equiv s_{\bar{M}}^{*2} + \left\langle s_M'^{2} \right\rangle \tag{II.25}$$

$$s_A^{o2} \equiv s_{\bar{A}}^{*2} + \left\langle s_A'^{2} \right\rangle \tag{II.26}$$

The space-time correlation can be also decomposed into

$$s_M^{o} s_A^{o} R_{MA}^{o} \equiv s_{\bar{M}}^{*} s_{\bar{A}}^{*} R_{\bar{M}\bar{A}}^{*} + \left\langle s_M' s_A' R_{MA}' \right\rangle \tag{II.27}$$

After substituting (II.25), (II.26) and (II.27) into (II.23), we obtain

$$\left\langle \overline{d_{MA}^2} \right\rangle = \left\langle \overline{d^2_{\langle\bar{M}\rangle\langle\bar{A}\rangle}} \right\rangle + s_{\bar{M}}^{*2} + s_{\bar{A}}^{*2} + \left\langle s_M'^{2} \right\rangle + \left\langle s_A'^{2} \right\rangle - 2 s_{\bar{M}}^{*} s_{\bar{A}}^{*} R_{\bar{M}\bar{A}}^{*} - 2\left\langle s_M' s_A' R_{MA}' \right\rangle \tag{II.28}$$

Equation (II.28) can be also obtained from equation (II.18) by successively performing the time and the spatial decomposition.

According to Boer and Lambert (2000), a perfect climate model should respect the following characteristics:

$$\left\langle \bar{M} \right\rangle = \left\langle \bar{A} \right\rangle$$

$$s_{\bar{M}}^{*} = s_{\bar{A}}^{*}$$

$$R_{\bar{M}\bar{A}}^{*} = 1$$

$$s_M' = s_A'$$

Since a climate model does not need to reproduce the temporal evolution of the weather systems, no condition is assigned to R'_{MA}. If these conditions are fulfilled, equation (II.28) reduces to

$$\left\langle \overline{d_{MA}^2} \right\rangle = \left\langle s_M'^{2} \right\rangle + \left\langle s_A'^{2} \right\rangle - 2\left\langle s_M' s_A' R_{MA}' \right\rangle \tag{II.29}$$

We can rewrite (II.29) as

$$\left\langle \overline{d_{MA}^2} \right\rangle = \left\langle s_M'^{2} \right\rangle + \left\langle s_A'^{2} \right\rangle - 2\left\langle s_M' s_A' R_{MA}' \right\rangle + 2\left\langle s_M' s_A' \right\rangle - 2\left\langle s_M' s_A' \right\rangle \tag{II.30}$$

which, after rearranging terms, becomes

$$\left\langle \overline{d_{MA}^2} \right\rangle = \left\langle \left(s_M' - s_A' \right)^2 \right\rangle + 2\left\langle s_M' s_A' \left(1 - R_{MA}' \right) \right\rangle \tag{II.31}$$

In a climatic application, two time series can be completely uncorrelated ($R'_{MA}=0$) but still possess identical time-averages and standard-deviations ($s_M' = s_A'$). In this case, (II.31) becomes

$$\left\langle \overline{d_{MA}^2} \right\rangle = 2\left\langle s_M' s_A' \right\rangle \neq 0 \tag{II.32}$$

Equation (II.32) shows that the usual definition of the space-time mean square difference is appropriate for a deterministic forecast (where $R'_{MA}=1$) but not for a climate projection because of this non-zero contribution. Boer and Lambert (2000) suggest removing this term by redefining the space-time mean square difference:

$$\left\langle \overline{\Delta^2_{MA}} \right\rangle = \left\langle \overline{d^2_{MA}} \right\rangle - 2\left\langle s'_M s'_A \left(1 - R'_{MA}\right)\right\rangle \qquad \text{(II.33)}$$

using (II.23), this expression can then be rewritten as

$$\left\langle \overline{\Delta^2_{MA}} \right\rangle = \left\langle \overline{d^2_{\langle\overline{M}\rangle\langle\overline{A}\rangle}} \right\rangle + s^{o2}_M + s^{o2}_A - 2 s^{o}_M s^{o}_A R^{o}_{MA} - 2\left\langle s'_M s'_A \left(1 - R'_{MA}\right)\right\rangle \qquad \text{(II.34)}$$

We can define an effective space-time correlation by combining the last two terms of (II.34), hence we can write

$$\left\langle \overline{\Delta^2_{MA}} \right\rangle = \left\langle \overline{d^2_{\langle\overline{M}\rangle\langle\overline{A}\rangle}} \right\rangle + s^{o2}_M + s^{o2}_A - 2 s^{o}_M s^{o}_A \hat{R}_{MA} \qquad \text{(II.35)}$$

where

$$\hat{R}_{MA} = R^{o}_{MA} + \frac{\left\langle s'_M s'_A \left(1 - R'_{MA}\right)\right\rangle}{s^{o}_M s^{o}_A} \qquad \text{(II.36)}$$

or in an alternate form

$$\hat{R}_{MA} = \frac{s^{*}_{\overline{M}} s^{*}_{\overline{A}} R^{*}_{\overline{M}\,\overline{A}}}{s^{o}_M s^{o}_A} + \frac{\left\langle s'_M s'_A \right\rangle}{s^{o}_M s^{o}_A} \qquad \text{(II.37)}$$

Again, the mean square difference between the spatial averages of the stationary terms can be removed from (II.35). After scaling the remaining contribution by the space-time variance of the observations, we finally obtain:

$$\frac{\left\langle \overline{\Delta^2_{MA}} \right\rangle_{BLT}}{s^{o2}_A} = 1 + \frac{s^{o2}_M}{s^{o2}_A} - \frac{2 s^{o}_M \hat{R}_{MA}}{s^{o}_A} \qquad \text{(II.38)}$$